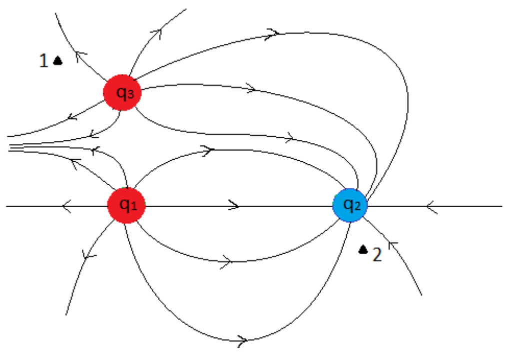
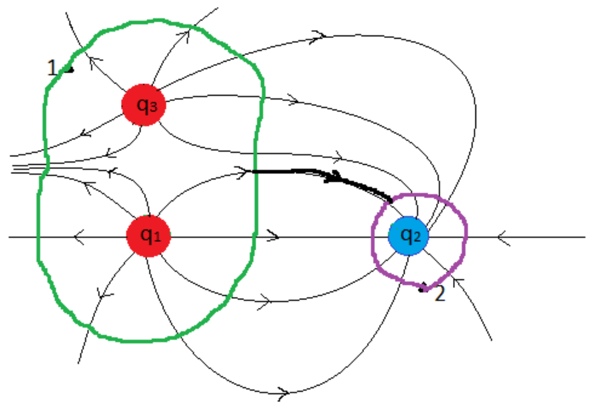
Homework 3 Solutions: Electric Potential

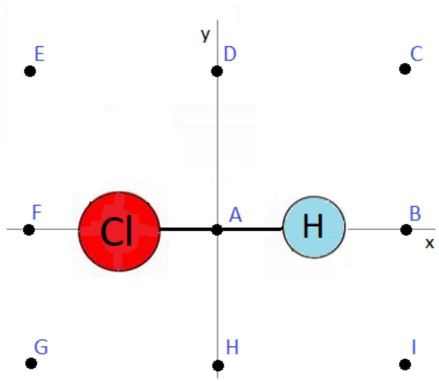
Problem 1. Consider the two points below. Draw the equipotential curve running through each. Then determine which is at the higher potential, and justify your answer by drawing a path between the equipotentials whch follows a field line.





Green equipotential is higher than purple one, since you follow a field line to get to it.

Problem 2. Let's reconsider the HCl molecule from before . Calculate the electric potential at each of the following points. Recall H has a charge +e, Cl a charge -e, and that the bond length is ℓ = 127pm. Answers should be in the 0 to $\pm$100 Volt range, when they're not zero. And place your answers next to the points in the figure, por favor.



(a) Point A = (0pm, 0pm).

So,

$$V = V_{Cl} + V_H = \frac{kq_{Cl}}{r_{Cl}} + \frac{kq_H}{r_H} = \frac{k(-e)}{63.5\text{pm}} + \frac{ke}{63.5\text{pm}} = 0$$

(b) Point B = (100pm, 0pm)?

And,

$$V = V_{Cl} + V_H = \frac{kq_{Cl}}{r_{Cl}} + \frac{kq_H}{r_H} = \frac{k(-e)}{(163.5\text{pm})} + \frac{ke}{(37.5\text{pm})} = 30\text{V}$$

(c) Point C = (100pm, 100pm)?

Yep,

$$V = V_{Cl} + V_H = \frac{kq_{Cl}}{r_{Cl}} + \frac{kq_H}{r_H} = \frac{k(-e)}{\sqrt{(163.5\text{pm})^2 + (100\text{pm})^2}} + \frac{ke}{\sqrt{(37.5\text{pm})^2 + (100\text{pm})^2}} = 6\text{V}$$

(d) Point D = (0pm, 100pm)?

So….

$$V = V_{Cl} + V_H = \frac{kq_{Cl}}{r_{Cl}} + \frac{kq_H}{r_H} = \frac{k(-e)}{\sqrt{(63.5\text{pm})^2 + (100\text{pm})^2}} + \frac{ke}{\sqrt{(63.5\text{pm})^2 + (100\text{pm})^2}} = 0$$

(e) Point E = (-100pm, 100pm)?

Could get this by symmetry, but I'll do it out….

$$V = V_{Cl} + V_H = \frac{kq_{Cl}}{r_{Cl}} + \frac{kq_H}{r_H} = \frac{k(-e)}{\sqrt{(37.5\text{pm})^2 + (100\text{pm})^2}} + \frac{ke}{\sqrt{(163.5\text{pm})^2 + (100\text{pm})^2}} = -6\text{V}$$

(f) Point F = (-100pm, 0pm)?

Could get this by symmetry too, but I'll do it out,

$$V = V_{Cl} + V_H = \frac{kq_{Cl}}{r_{Cl}} + \frac{kq_H}{r_H} = \frac{k(-e)}{37.5\text{pm}} + \frac{ke}{163.5\text{pm}} = -30\text{V}$$

(g) Point G = (-100pm,-100pm)?

Again, could do symmetry, but:

$$V = V_{Cl} + V_H = \frac{kq_{Cl}}{r_{Cl}} + \frac{kq_H}{r_H} = \frac{k(-e)}{\sqrt{(37.5\text{pm})^2 + (100\text{pm})^2}} + \frac{ke}{\sqrt{(163.5\text{pm})^2 + (100\text{pm})^2}} = -6\text{V}$$

(h) Point H = (0pm, -100pm)?

Symmetry, or….

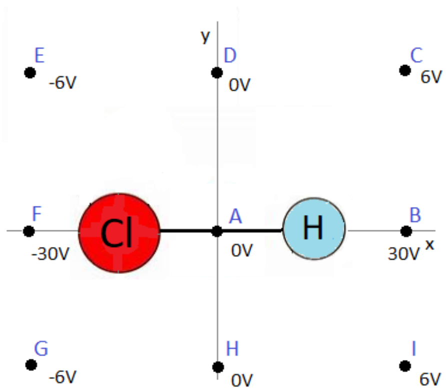
$$V = V_{Cl} + V_H = \frac{kq_{Cl}}{r_{Cl}} + \frac{kq_H}{r_H} = \frac{k(-e)}{\sqrt{(63.5\text{pm})^2 + (100\text{pm})^2}} + \frac{ke}{\sqrt{(63.5\text{pm})^2 + (100\text{pm})^2}} = 0$$

(i) Point I = (100pm, -100pm)?

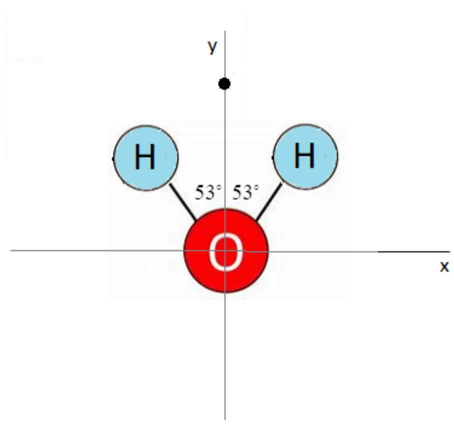
Symmetry, or….

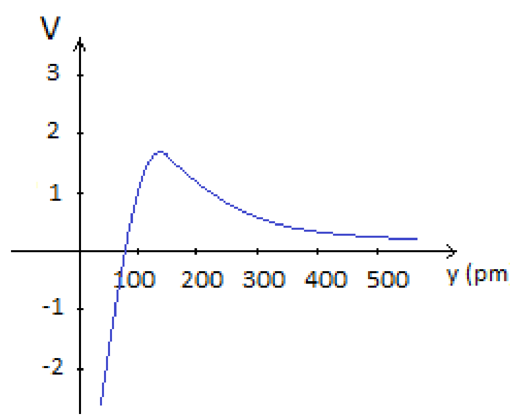
$$V = V_{Cl} + V_H = \frac{kq_{Cl}}{r_{Cl}} + \frac{kq_H}{r_H} = \frac{k(-e)}{\sqrt{(163.5\text{pm})^2 + (100\text{pm})^2}} + \frac{ke}{\sqrt{(37.5\text{pm})^2 + (100\text{pm})^2}} = 6\text{V}$$

So this is what we get:



Problem 3. Remember the water molecule, where H has an effective charge of +0.35e, and O and effective charge of -0.70e, and bond length 100pm? Calculate the electric potential of the water molecule's field, along the y axis. And plot it below.

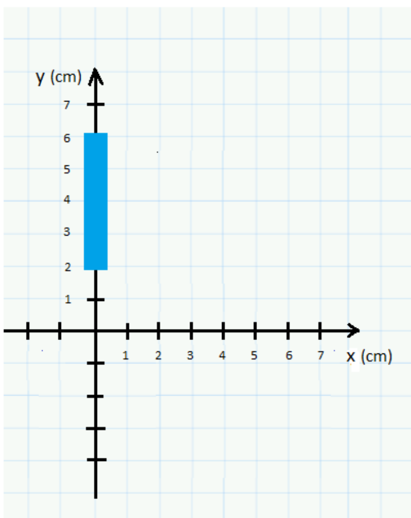




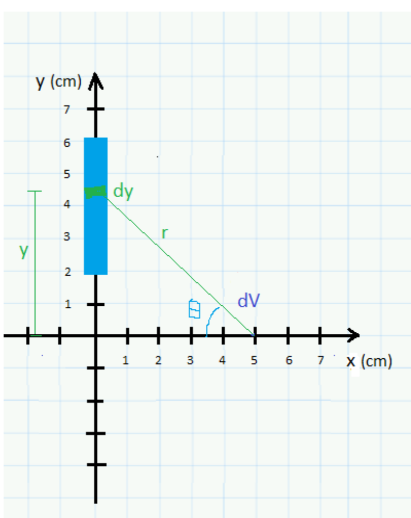
Welp,

$$
\begin{aligned}
V(y) &= V_O + V_H + V_H \\
&= \frac{k(-0.70e)}{y} + \frac{k(0.35e)}{\sqrt{(y-60\text{pm})^2+(80\text{pm})^2}} + \frac{k(0.35e)}{\sqrt{(y-60\text{pm})^2+(80\text{pm})^2}} \\
&= \frac{k(-0.70e)}{y} + \frac{k(0.70e)}{\sqrt{(y-60\text{pm})^2+(80\text{pm})^2}} \\
&= k(0.70e)\left[\frac{1}{\sqrt{(y-60\text{pm})^2+(80\text{pm})^2}} - \frac{1}{y}\right]
\end{aligned}
$$

Problem 4. Remember this guy? Consider a plastic rod charged non-uniformly as λ(y) = -2y (nC/m).
(a) Calculate its electric potential at the point x = 5cm. Answer should be in the half Volt range.



So if we single out a piece of charge dq, and calculate its potential dV, and then integrate over all dV's we'd get:



$$V = \int dV \qquad \begin{cases} dV = \dfrac{kdq}{r} \\ dq = \lambda dy = -2ydy\,(\text{nC/m}) = -2\times10^{-9}ydy \\ r = \sqrt{y^2 + 0.05^2} \end{cases}$$

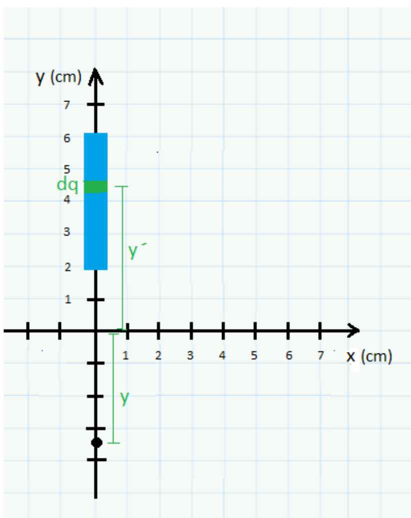
$$= \int_{0.02}^{0.06} \frac{(9\times10^9)(-2\times10^{-9})ydy}{\sqrt{y^2+0.05^2}}$$

$$= -18\int_{0.02}^{0.06} \frac{ydy}{(y^2+0.05^2)^{1/2}}$$

$$= -0.44\,\text{V}$$

(b) Give an expression for the electric potential at any coordinate y below the origin. If you do it right, then you should get V(-1·meter) = -28mV, for instance.

So this would be given by:



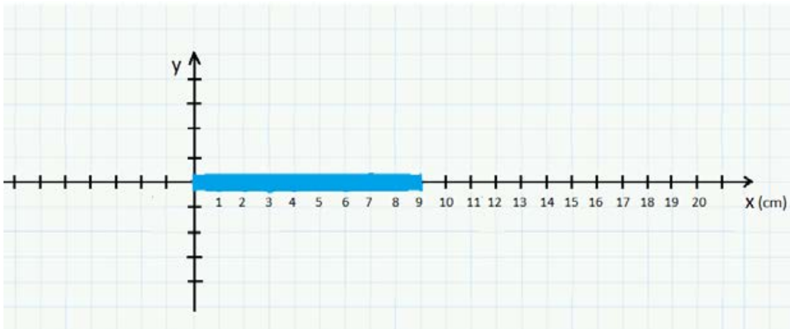
$$V = \int dV \qquad \begin{cases} dV = \dfrac{kdq}{r} \\ dq = \lambda dy' = -2y'dy'\,(\text{nC/m}) = -2\times10^{-9}y'dy' \\ r = y' - y \end{cases}$$

$$= \int_{0.02}^{0.06} \frac{(9\times10^9)(2\times10^{-9})y'dy'}{y'-y}$$

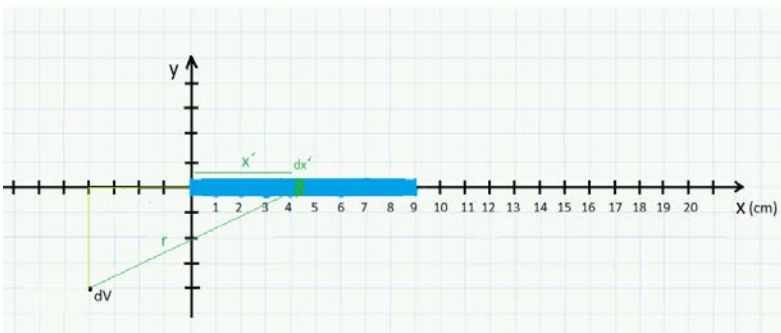
$$= -18\int_{0.02}^{0.06} \frac{y'dy'}{y'-y}$$

$$= 18y\ln\left(\frac{0.02-y}{0.06-y}\right) - 0.72$$

Problem 5. Now consider that plastic rod from before, charged uniformly with 10nC. Give an expression for the electric potential at any point (x,y) in the third quadrant. If you do this right, then you should get V(-1,-1) = 62V, for instance.



So we isolate a piece of charge, calculate dV, and integrate…



$$V = \int dV \qquad \begin{cases} dV = \dfrac{kdq}{r} \\ dq = (1.11\times10^{-7}\text{ C/m})dx \\ r = \sqrt{(x'-x)^2 + y^2} \end{cases}$$

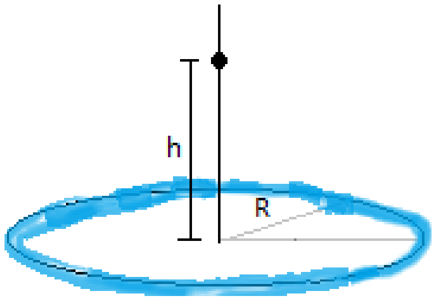
$$= -\int_0^{0.09} \frac{(9\times10^9)(1.11\times10^{-7})dx'}{\sqrt{(x'-x)^2+y^2}}$$

$$= -1000\int_0^{0.09} \frac{dx'}{\sqrt{(x'-x)^2+y^2}}$$

$$= 1000\ln\left[\sqrt{(x-x')^2+y^2} + x' - x\right]\Bigg|_{x=0}^{x=.09}$$

$$= 1000\ln\left[\frac{\sqrt{(x-0.09)^2+y^2}+0.09-x}{\sqrt{x^2+y^2}-x}\right]$$

Problem 6. Say we have the same R = 3m ring as last time, and that we again smear 7nC of charge on it, but this time uniformly over the entire ring. Give an expression for the potential at any height h. If you do this right then you should get V(h = 1·meter) = 20V.



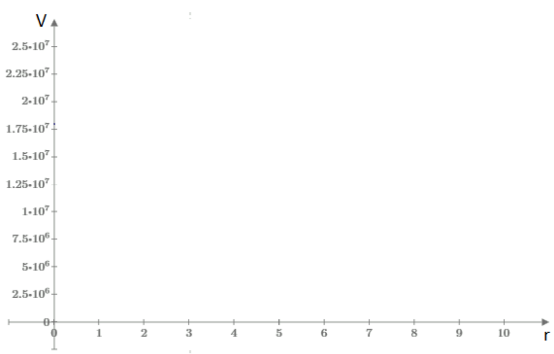
This one's easy….

$$V=\int dV=\int\frac{kdq}{r}=\int\frac{kdq}{\sqrt{3^2+h^2}}=\frac{k}{\sqrt{3^2+h^2}}\int dq=\frac{kq}{\sqrt{3^2+h^2}}=\frac{k(7\times10^{-9})}{\sqrt{3^2+h^2}}=\frac{63}{\sqrt{3^2+h^2}}$$

Problem 7. Now let's go back to that dust cloud problem. Recall that it had a 3m radius, and that its field was (hopefully) this thing below. Make a plot of the cloud's electric potential as a function of radius, taking r = ∞ as your reference point. Should get V(0) as about 18MV.

So,

$$E=\frac{kq_{enclosed}}{r^2}=\begin{cases}1.33\times10^{6}r & \text{inside}\\ \dfrac{36\times10^{6}}{r^2} & \text{outside}\end{cases}$$



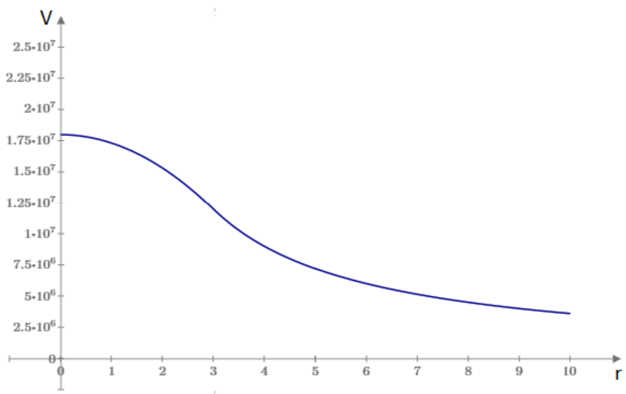
So potential is given by:

$$V = -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{r}$$

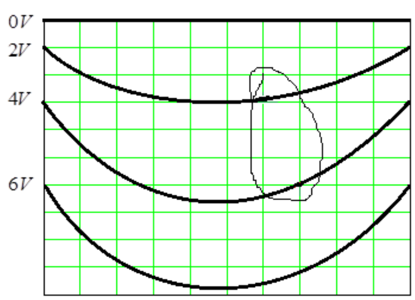
$$= \begin{cases} -\int_{\infty}^{r} \frac{36\times10^6}{r^2}\,dr & \text{outside} \\ -\int_{\infty}^{3} \frac{36\times10^6}{r^2}\,dr - \int_{3}^{r} 1.33\times10^6\, r\,dr & \text{inside} \end{cases}$$

$$= \begin{cases} \frac{36\times10^6}{r} & \text{outside} \\ \frac{36\times10^6}{3} - 1.33\times10^6\left(\frac{r^2}{2} - \frac{3^2}{2}\right) & \text{inside} \end{cases}$$

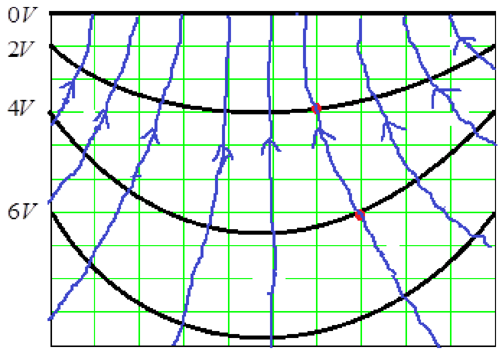
And I get:



Problem 8. Consider the equipotentials displayed on the grid below. Draw in five electric field lines (including arrows) that would correspond to them. And estimate the field strength in the circled region. Squares are 1cm×1cm.



Field lines look like this, perpendicular to equipotentials, and running towards lower potential.



The approximate field strength for the line in particular, between the 2V and 6V equipotentials would be:

$$E = \left|\frac{\Delta V}{\Delta s}\right| = \frac{4V - 2V}{\sqrt{(0.01\text{m})^2 + (0.03\text{m})^2}} = 63\,\text{V/m}$$

Problem 9. Going back to problem 2. Use your plot to determine where the electric field is pointing left, where it is zero, and where it is pointing to the right.
So we can see, from $E_y = -dV/dy$, that it is pointing to the left before y ≈ 130pm, it's zero at y ≈ 130pm, and it's pointing to the right afterwards. Note this all agrees with HW 1.

Problem 10. Going back to problem 3b, use your V(y) expression to calculate the electric field strength and direction at y = -3cm. You should get what you calculated in HW 1.
So,

$$V(y) = 18y\ln\left(\frac{0.02-y}{0.06-y}\right) - 0.72$$

$$E_y = -\frac{dV}{dy} = -\frac{d}{dy}\left[18y\ln\left(\frac{0.02-y}{0.06-y}\right) - 0.72\right]$$

$$= -18\ln\left(\frac{0.02-y}{0.06-y}\right) - 18y\cdot\left[\frac{-1}{0.02-y} - \frac{-1}{0.06-y}\right]$$

$$= -18\ln\left(\frac{0.02-y}{0.06-y}\right) + \frac{0.72y}{(0.02-y)(0.06-y)}$$

Plugging in y = -0.03, we get 5.8N/C, which does indeed agree.

Problem 11. Going back to problem 4, use your V(x,y) expression to calculate the electric field strength and direction at the coordinate (-4cm, -4cm). You should get what you calculated in HW 1.
Indeed?

$$V(x,y) = 1000\ln\left[\frac{\sqrt{(x-0.09)^2+y^2}+0.09-x}{\sqrt{x^2+y^2}-x}\right]$$

$$\begin{aligned}
\mathbf{E} &= -\nabla V \\
&= -\frac{dV}{dx}\hat{\mathbf{i}} - \frac{dV}{dy}\hat{\mathbf{j}} \\
&= -1000\left\{\frac{\frac{x-0.09}{\sqrt{(x-0.09)^2+y^2}}-1}{\sqrt{(x-0.09)^2+y^2}+0.09-x} - \frac{\frac{x}{\sqrt{x^2+y^2}}-1}{\sqrt{x^2+y^2}-x}\right\}\hat{\mathbf{i}} - 1000\left\{\frac{\frac{y}{\sqrt{(x-0.09)^2+y^2}}}{\sqrt{(x-0.09)^2+y^2}+0.09-x} - \frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}-x}\right\}\hat{\mathbf{j}}
\end{aligned}$$

Yeah it ain't pretty. But I guess that's life. So now plug in the coordinate x = -0.04, y = -0.04, and we get:

$$\begin{aligned}
\mathbf{E} &= -10000\hat{\mathbf{i}} - 6200\hat{\mathbf{j}} \\
&= \sqrt{(10300)^2+(6220)^2} \,@\, \tan^{-1}\left(\frac{-6220}{-10300}\right)^{\circ} = 12000\text{ N/C} \,@\, 31^{\circ} \text{ below } -x \text{ axis}
\end{aligned}$$

which matches our result from HW 1.

Problem 12. Last one. Reconsider that ring in problem in problem 5. Use your V(y) formula to get a formula for the electric field at any height y above the ring.
Ahhh, a nice relaxing one-line problem to finish everything off.

$$V(y) = \frac{63}{\sqrt{3^2+y^2}}$$

$$E_y = -\frac{dV}{dy} = -\frac{d}{dy}\frac{63}{\sqrt{3^2+y^2}} = \frac{63y}{(3^2+y^2)^{3/2}}$$